

A symmetry of the spatially flat Friedmann equations with barotropic fluids

Valerio Faraoni

*Physics Department and STAR Research Cluster, Bishop's University, 2600 College
Street, Sherbrooke, Québec, Canada J1M 1Z7*

Abstract

A string-inspired duality symmetry of the spatially flat Friedmann equations of general-relativistic cosmology is discussed and generalized, providing a map between exact solutions corresponding to different values of the barotropic index.

Keywords: Friedmann cosmology, duality

1. Introduction

In modern theoretical physics, the symmetry group of a theory plays a crucial role and one of the first questions asked of a new theory is which are its symmetries. Here we point out a symmetry property of the Einstein-Friedmann equations, the fundamental equations of spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology, for the case of flat spatial sections. A special subcase has been discussed extensively in the literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], and it very much resembles a well known duality of string cosmology. In spite of a significant amount of literature on the symmetries of the Friedmann equations, with particular attention to generating new analytical solutions from known ones, in both classical and quantum cosmology ([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and references therein), to the best of our knowledge the general symmetry has not been commented upon¹.

¹See also [21] for a solution of the Einstein-Friedmann equations (which, for a barotropic fluid, reduce to a well known Riccati equation [22]) using methods of supersymmetric

We restrict ourselves to spatially flat FLRW cosmology and to the line element

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2) \quad (1)$$

in comoving coordinates (t, x, y, z) . The scale factor is taken to be dimensionless, while the coordinates carry the dimension of a length. We assume a perfect fluid as the matter source in the Einstein equations of general relativity, which reduce to [23, 24, 25]

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} (\rho + 3P) , \quad (2)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} \rho , \quad (3)$$

where $\rho(t)$ and $P(t)$ are the energy density and pressure of the perfect fluid, respectively, and we use units in which Newton's constant G and the speed of light are unity. The perfect fluid is described by the stress-energy tensor

$$T_{ab} = (P + \rho) u_a u_b + P g_{ab} , \quad (4)$$

where u^a is the fluid (timelike) four-velocity which has components $u^\mu = \delta_0^\mu$ in comoving coordinates. We further assume that the perfect fluid is ruled by the barotropic equation of state

$$P = w\rho \equiv (\gamma - 1) \rho , \quad (5)$$

where the barotropic index γ is constant and w is usually referred to as the equation of state parameter. As is well known, the covariant conservation equation $\nabla^b T_{ab} = 0$ yields

$$\dot{\rho} + 3H (P + \rho) = 0 , \quad (6)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. This conservation equation can also be obtained directly from eqs. (2) and (3) and is immediately integrated yielding ρ as a function of the scale factor,

$$\rho(a) = \frac{\rho_0}{a^{3\gamma}} = \frac{\rho_0}{a^{3(w+1)}} . \quad (7)$$

quantum mechanics.

2. A symmetry of the Einstein-Friedmann equations

Let us perform the operation

$$a \rightarrow \alpha \equiv \frac{1}{a}; \quad (8)$$

then $H = -\dot{\alpha}/\alpha$, while $\frac{\ddot{a}}{a} = 2\left(\frac{\dot{\alpha}}{\alpha}\right)^2 - \frac{\ddot{\alpha}}{\alpha}$ and eqs. (2) and (3) become

$$\frac{\ddot{\alpha}}{\alpha} = -\frac{4\pi}{3}(3\tilde{\gamma} - 2)\rho, \quad (9)$$

$$\left(\frac{\dot{\alpha}}{\alpha}\right)^2 = \frac{8\pi}{3}\rho, \quad (10)$$

where $\tilde{\gamma} \equiv -\gamma$. That is, eqs. (2) and (3) are invariant in form under the transformation

$$a \rightarrow \alpha \equiv \frac{1}{a}, \quad \gamma \rightarrow \tilde{\gamma} = -\gamma. \quad (11)$$

The inversion of the scale factor can be compensated by the change in the equation of state

$$P = w\rho \rightarrow P = \tilde{w}\rho = -(w + 2)\rho \quad (12)$$

where $\tilde{w} = -(w + 2)$ (equivalent to $\tilde{\gamma} = -\gamma$). Therefore, there is a duality in the equations ruling spatially flat FLRW cosmology which maps expanding universes into contracting ones, and *vice-versa*. Assuming that the barotropic index $\gamma \neq 0$ and that the weak energy condition ($\rho \geq 0$ and $\rho + P = \gamma\rho \geq 0$) [24] is satisfied by a universe with scale factor $a(t)$ and perfect fluid equation of state (5), the “dual universe” with scale factor $\alpha(t) = 1/a(t)$ certainly violates it since $\tilde{\gamma}\rho = -\gamma\rho < 0$. The perfect fluid associated with the dual universe is, for sure, a phantom fluid. For example, the dual of a stiff fluid universe with $P = \rho$ is a strongly phantom fluid with $P = -3\rho$; this situation can be realized by a massless scalar field with no potential. In fact, the energy density and pressure of a minimally coupled scalar are

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi) \quad (13)$$

and

$$P = \frac{\dot{\phi}^2}{2} - V(\phi); \quad (14)$$

setting $V \equiv 0$ yields $P = \rho$ and it is well known that in the FLRW space a minimally coupled scalar field can be given a fluid description [26].

The conservation equation (6), of course, does not change by virtue of being a consequence of the field equations and not an independent equation. It is straightforward to check directly, using eqs. (9) and (10), that

$$\dot{\rho} + 3 \frac{\dot{\alpha}}{\alpha} \tilde{\gamma} \rho = 0 \quad (15)$$

the solution of which is, of course, $\rho(\alpha) = \rho_0/\alpha^{3\gamma}$ and can be obtained directly from eq. (7).

For $\gamma = \text{const.} \neq 0$, the solution of eqs. (2) and (3) is the power-law

$$a(t) = a_0 |t - t_*|^{\frac{1}{3\gamma}} = a_0 |t - t_*|^{\frac{1}{3(w+1)}} \quad (16)$$

as is well known. For $\gamma > 0$ and $t > t_*$ this is a Big Bang universe with initial singularity at t_* . The “dual” solution² is given by

$$\alpha(t) = \frac{\alpha_0}{|t - t_*|^{\frac{1}{3\gamma}}} \quad (17)$$

and is a contracting universe beginning from an $a = \infty$ singularity at t_* , as a consequence of the fact that the fluid with $P + \rho = \gamma\rho$ has changed into a phantom fluid with $\tilde{\gamma}\rho = -\gamma\rho$ in the dual solution. For $\gamma > 0$ and $t < t_*$ there is a duality between a contracting universe ending in a Big Crunch singularity at t_* and an expanding superaccelerating (*i.e.*, $\dot{H} > 0$) universe ending in a Big Rip singularity with $a = \infty$ at t_* .

For $\gamma = 0$, the solution of eqs. (2) and (3) is the expanding de Sitter space $a(t) = a_0 e^{Ht}$ ($H = \text{const.} > 0$) and the “dual” universe is the contracting de Sitter space $\alpha(t) = \alpha_0 e^{-Ht}$. Minkowski space (the degenerate case with $H = 0$ and $\rho = P = 0$) is obviously a fixed point of the transformation (11), but a trivial one.

The duality symmetry has been extended to brane world models with bulk effects switched off in [3] (see also [27]).

²Since the duality transformation does not involve explicitly the time t , it acts only in one of the two branches $t < t_*$ and $t > t_*$.

3. Generalizing the duality

Since the Einstein-Friedmann equations are so fundamental and in view of the extensive applications of the duality symmetry introduced in the previous section to classical and quantum cosmology [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], it is useful to generalize it by asking what is required in order to map a perfect fluid solution of eqs. (2) and (3) with constant barotropic index γ_1 into a solution with index γ_2 (with $\gamma_1, \gamma_2 \neq 0$). All FLRW metrics are conformally flat, and the spatially flat FLRW line element is explicitly conformal to the Minkowski one $ds_0^2 = -d\eta^2 + dx^2 + dy^2 + dz^2$. Using the conformal time η defined by $dt = a d\eta$ we have,³ for the two FLRW solutions,

$$ds_1^2 = -dt^2 + a_1^2(t) (dx^2 + dy^2 + dz^2) = \Omega_1^2 ds_0^2, \quad (18)$$

$$ds_2^2 = -dt^2 + a_2^2(t) (dx^2 + dy^2 + dz^2) = \Omega_2^2 ds_0^2, \quad (19)$$

where $\Omega_{1,2} = a_{1,2}(\eta)$ is the conformal factor for each case, and

$$a_1(t) = a_* |t - t_*|^{\frac{1}{3\gamma_1}}, \quad (20)$$

$$a_2(t) = a_* |t - t_*|^{\frac{1}{3\gamma_2}} \quad (21)$$

(the constant a_* needs not be the same in both cases, but it is irrelevant because it can always be eliminated by rescaling the coordinates). Clearly, it is

$$ds_2^2 = \Omega_2^2 ds_0^2 = \Omega^2 ds_1^2, \quad (22)$$

where

$$\Omega = \frac{\Omega_2}{\Omega_1} = \frac{a_2}{a_1} = \text{const.} \cdot |t - t_*|^{\frac{\gamma_2 - \gamma_1}{\gamma_1(3\gamma_2 - 1)}} \quad (23)$$

($\gamma_2 \neq 1/3$). In other words, we can obtain a solution of eqs. (2) and (3) simply by conformally transforming another solution, because they are both conformal to Minkowski space and, therefore, are conformally related. Of

³For the solution given by eq. (16), the conformal time is $\eta(t) = \frac{3\gamma}{(3\gamma-1)a_0} |t - t_*|^{\frac{1-3\gamma}{3\gamma}}$ and $a(\eta) = \left(\frac{3\gamma}{3\gamma-1}\right)^{\frac{1}{3\gamma-1}} a_0 \eta^{\frac{1}{3\gamma-1}}$.

course, one could simply obtain $|t - t_*|$ from a_1 and substitute it into a_2 to obtain $a_2 = a_1^{\gamma_1/\gamma_2}$, without the need to identify conformal transformations, but then the discussion would not be generally covariant.

The transformation Γ_{12} described by

$$\gamma_1 \longrightarrow \gamma_2 \quad \left(\gamma_1 \neq 0, \quad \gamma_2 \neq 0, \frac{1}{3} \right), \quad (24)$$

$$a_1(t) \longrightarrow a_2(t) = [a_1(t)]^{\gamma_1/\gamma_2}, \quad (25)$$

is a symmetry of the Einstein-Friedmann equations (2) and (3). This property can be checked explicitly by substituting the expression $a_2 = A a_1^{\gamma_1/\gamma_2}$ (where A is a constant) into eqs. (2) and (3). Eqs. (2) and (3) are satisfied if, given the energy density $\rho_1 = \rho_1^{(0)}/a_1^{3\gamma_1}$, it is

$$\rho_1 \longrightarrow \rho_2 = \left(\frac{\gamma_1}{\gamma_2} \right)^2 \rho_1, \quad (26)$$

while

$$H_1 \equiv \frac{\dot{a}_1}{a_1} \longrightarrow H_2 \equiv \frac{\dot{a}_2}{a_2} = \frac{\gamma_1}{\gamma_2} H_1. \quad (27)$$

The transformations Γ_{ij} form a non-commutative group with respect to the operation of composition of transformations \circ . In fact, $\Gamma_{jk} \circ \Gamma_{ij} = \Gamma_{ik}$, the zero element is the identity Γ_{ii} , and the inverse of Γ_{ij} is $\Gamma_{ij}^{-1} = \Gamma_{ji}$.

As a special case, one recovers the map of the previous section which has been studied in the literature

$$\gamma_1 \equiv \gamma \longrightarrow \gamma_2 = -\gamma, \quad (28)$$

for which $a_2 = a_1^{\gamma_1/\gamma_2} = 1/a$.

4. Discussion

The duality $(a, \gamma) \longleftrightarrow (a^{-1}, -\gamma)$ is used in a variety of applications to generate new solutions from old ones and it is of practical utility. It is interesting to try to understand if this symmetry comes from more fundamental theories which leave a memory in the limit to general-relativistic cosmology (keeping in mind that this is not a symmetry of the full Einstein equations,

nor the well known time-reversal symmetry). The duality discussed here resembles a duality in the spatially flat FLRW space of pre-big bang theories [28, 29, 30, 31], which is generalized by another symmetry in the spatially flat FLRW cosmology of vacuum Brans-Dicke theory [32]. Restricting ourselves to the purely gravitational sector of the theory, the Brans-Dicke field ϕ still acts as an effective form of matter, producing the equations of motion [32]

$$\dot{H} = -\frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi} \right)^2 + 2H \frac{\dot{\phi}}{\phi} + \frac{1}{2(2\omega + 3)} \left(\phi \frac{dV}{d\phi} - 2V \right), \quad (29)$$

$$H^2 = \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi} \right)^2 - H \frac{\dot{\phi}}{\phi} + \frac{V(\phi)}{6\phi}, \quad (30)$$

where $V(\phi)$ is the scalar field potential (absent in the original Brans-Dicke formulation but naturally present in high energy theories) and ω is the Brans-Dicke parameter [32]. In the absence of matter, the Brans-Dicke field satisfies the equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega + 3} \left(-\phi \frac{dV}{d\phi} + 2V \right) = 0. \quad (31)$$

It is well known that vacuum, spatially flat FLRW cosmology enjoys a duality symmetry [33, 34]. Let us use new variables

$$\beta \equiv \ln a, \quad \Phi \equiv -\ln(G\phi) \quad (32)$$

instead of a and ϕ ; then the duality transformation takes the form [33, 34]

$$\beta \rightarrow \left(\frac{3\omega + 2}{3\omega + 4} \right) \beta - 2 \left(\frac{\omega + 1}{3\omega + 4} \right) \Phi, \quad (33)$$

$$\Phi \rightarrow \left(-\frac{6}{3\omega + 4} \right) \beta - \left(\frac{3\omega + 2}{3\omega + 4} \right) \Phi, \quad (34)$$

for $\omega \neq -4/3$. This transformation generalizes the duality present in the effective action of string theories [28, 29, 30, 31]

$$\beta \rightarrow -\beta, \quad (35)$$

$$\Phi \rightarrow \Phi - 6\beta, \quad (36)$$

which is reproduced by eqs. (33) and (34) for $\omega = -1$ (remember that the bosonic string theory reduces to a Brans-Dicke theory with this value of the Brans-Dicke parameter [35]). In general relativity, the effective gravitational coupling corresponding to ϕ^{-1} is a constant, not a dynamical variable, and eq. (36) would not make sense there; eq. (35) corresponds to the transformation (8). However, to make this transformation a true symmetry, it is required that a perfect barotropic fluid with constant equation of state be present, and that the transformation $\gamma \rightarrow -\gamma$ of this fluid accompanies (8). By contrast, string theory enjoys the symmetry (35) and (36) *in vacuo* (but Φ acts as a form of effective matter). In general, the Φ -field does not behave as a perfect fluid with constant equation of state due to its dynamical nature, which causes the effective parameter w (or γ) to be time-dependent. Therefore, in spite of the coincidence of eqs. (8) and (35), we conclude that there is no direct link between the symmetry of spatially flat FLRW space in general relativity and the symmetry (35), (36) of pre-big bang theories. Similarly, in the $\omega \rightarrow \infty$ limit of Brans-Dicke theory in which general relativity is (usually but not always⁴) reproduced, eqs. (33) and (34) do not reproduce the symmetry (11).

The duality $(a, \gamma) \longleftrightarrow (a^{-1}, -\gamma)$ turns out to be only a special case of the more general group of symmetries $(a_1, \gamma_1) \longleftrightarrow (a_1^{\gamma_1/\gamma_2}, \gamma_2)$; the application of this more general duality to classical cosmological scenarios and to the Wheeler-DeWitt equation will be the subject of future publications.

This work is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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⁴Situations in which this limit does not reproduce general relativity have been discussed in [36].

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